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The Association of Engineering and Shipbuilding Draughtsmen.

THE MOMENT DISTRIBUTION METHOD FOR CONTINUOUS BEAMS AND FRAMED STRUCTURES.

By BASIL STANDING,
M.Sc., A.M.I.Mech.E., M.R.San.I.

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"Moment Distribution Method for Continuous Beams and Framed Structures."

ERRATA.

Page 9, Fig. 7. A $-^{ve}$ and not a $+^{ve}$ sign should prefix the right-hand figure of 9·37 in the 3rd distribution.

Page 9, Fig. 9. The loads causing the F.E.M.s of 40 in the left-hand span are 9 lb. loads at the third points of the span.

Page 42. For the first type of loading given for M_L read M_R and vice-versa.

For the fourth type of loading given, for $6\,a\,l^2$ read $6\,l^2$.

Page 43. For the second type of loading, the formula given is for where half a unit load is carried at the supports, *i.e.*, where there are $n-1$ loads of P/n symmetrically spaced, and two loads of $P/_{2n}$, one at each support. Where there are only $n-1$ equal loads, the span being divided into n equal panels then

$$M_L = M_R = \frac{Pl}{12}\left(\frac{n-1}{n}\right)$$

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INTRODUCTION.

UP to the time of the introduction of the method of moment distribution by Professor Hardy Cross in 1930, the methods available for analysis of continuous beams with unsymmetrical spans and loadings, and of frameworks with rigid joints, were such that, in all but the simplest cases, the numerical calculations involved were extremely lengthy and laborious. The theorem of three moments applied to beams of varying section, and of more than about four spans, and especially if settlement of the supports occurred, involved the solution of simultaneous equations with cumbersome numerical coefficients. The method of slope deflection suffered from much the same defects, with the added complication of a sign-convention which was somewhat difficult of application. Strain-energy methods, with or without the use of Maxwell's theorem, had similar disadvantages, and the method of characteristic points required the use of quite complicated diagrams for even relatively simple frames.

In addition, these methods gave results which were not of as high a degree of accuracy as the amount of calculation implied, and which, in any case, it was impossible to attain, in view of the uncertainty in the elastic and other properties of the materials concerned. For example, the uncertainty in the exact value of E for concrete, or the neglect of the strain-energy due to shear when using Castigliano's theorems, meant that any great degree of refinement in the calculations was entirely unwarranted as far as the designer was concerned.

The method of moment distribution, whilst involving a certain amount of numerical computation, but of the simplest kind, has, over these previous methods, certain definite advantages :—

(*a*) No solution of equations is involved.
(*b*) The process can be made a purely mechanical one, capable of quick and easy checking.
(*c*) The accuracy obtained can be of any requisite degree.
(*d*) The method enables the deflected form of the structure to be readily visualised.

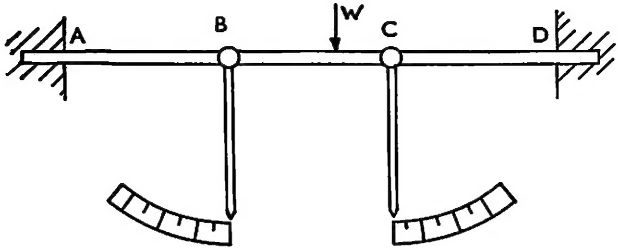


Fig. 1.

Method of Moment Distribution.

Imagine first of all a structure, as shown in Fig. 1, with a pointer and scale attached to each of the joints B and C. Joints A and D are "built-in," and therefore permanently fixed, but B and C may be prevented from rotation by clamps, or alternatively be free to rotate as desired. To start with, imagine B and C to be clamped, and let some load W be placed on span BC. The pointers will not move, as B and C are fixed, and the beam will be as in Fig. 2. Now let the clamp on joint B be loosened, and it may be imagined that the configuration will be as in Fig. 3, the pointer attached to B recording the rotation. Thus, with B clamped, a moment was caused at B by the load on BC, and the unclamping

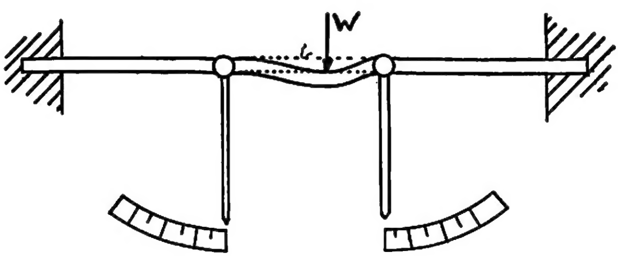


Fig. 2.

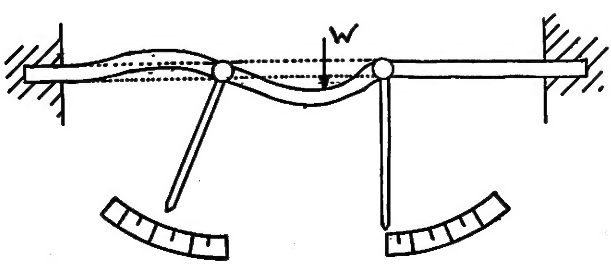


Fig. 3.

of B has now allowed span AB to be affected by the load, *i.e.*, to incur bending moments due to the load. Thus the moment at B, initially prevented from being transmitted to AB by the clamping of B, has been "distributed" to AB, or the "unbalanced" moment at B has been "distributed" or "balanced." Just how much moment has gone to BA, and how much has remained in BC will be investigated later, but for the moment it can be imagined that, from the theory of simple bending, these amounts will be proportional to both the second moments of area (I) of the sections of AB and BC, and to their span lengths (l).

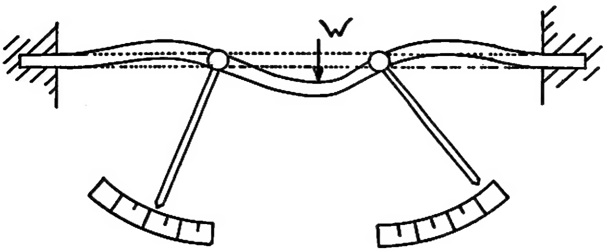


Fig. 4.

Now let B be clamped, and C released, and the pointer at C will indicate some such movement of C, as shown in Fig. 4. The moment at C has now been distributed. Now let C be clamped, and B released again. B will turn, as shown by the pointer, but not through as great a distance as before. But since B was balanced when it was released for the first time, this unbalanced moment, now being released, must have arisen at B as a result of the previous balancing at C. Thus it seems that a distribution at C, which divides an unbalanced moment there between the two ends of the spans (BC and CD) meeting there, also means that some moment is transmitted to B, *i.e.*, the part of a moment distributed to end C of span BC, also sets up a moment at the end B of BC. In other words, a certain amount of moment is "carried over" to the end of the span other than that at which it was applied.

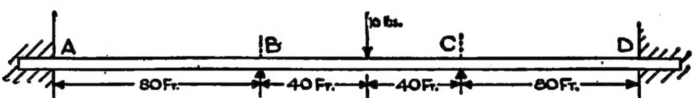
This process can now be continued, joints B and C being clamped and released alternately, till the motion of the joints, as recorded by the pointers, is imperceptible. The structure is now in equilibrium under the load, and the clamps may be removed. This procedure, in which a loaded structure is allowed to take up its normal deflected form in controlled stages, commencing with all members fixed at the ends, and in which each joint is released and fixed again in turn, so that unbalanced moments are distributed, is the basis of the method of moment distribution.

As an example, the method is applied to the three-span continuous beam just considered, with numerical values given to the

load, span, length, etc. The spans AB, BC and CD will be taken as each equal to 80 ft., and the values of I and E as constant for the three spans. Thus, ignoring the actual meaning of "stiffness" for the moment, the symmetry will ensure that an unbalanced moment will be distributed in equal parts on either side of a joint. If now B and C are clamped, corresponding to Fig. 2, and $W = 10$ lbs. is placed midway along BC, then the moments at B and C in BC will be the same as for a built-in beam of the same span, and under an identical load,

$$i.e., \quad \frac{Wl}{8} = \frac{10 \times 80}{8} = 100 \text{ lbs. ft.}$$

Columns are drawn as in Fig. 5, writing these moments as in line 1. They are the "fixed-end moments" (F.E.M.s), and will be both taken as $-^{ve}$. Now imagine joint B to be freed, so that it takes up a position of equilibrium with the moments on either side of it equal. This can only be done by distributing the unbalanced moment of -100, and, since AB and BC are of equal stiffness the moments will be distributed as in line 2, 50 units to each side of the joint. The signs of the distributed portions will be as shown, and as there is only one way of allotting signs to these portions so that the moments on either side of the joint are equal in magnitude and sign, there is no ambiguity, and no confusion as to sign should exist, and no errors should thereby be introduced.



	A	B	B	C	C	D	
LINE 1			-100	-100			F.E.M.s
" 2		-50	+50	+50	-50		1st Distrib.
" 3	+25		-25	-25		+25	Carry-Over.
" 4		-12·50	+12·50	+12·50	-12·50		2nd Distrib.
" 5	+6·25		-6·25	-6·25		+6·25	Carry-Over.
" 6		-3·12	+3·12	+3·12	-3·12		3rd Distrib.
" 7	+1·56		-1·56	-1·56		+1·56	Carry-Over.
" 8	+·39	-·78	+·78	+·78	-·78	+·39	4th Distrib.
FINAL MOMENTS	+33·20	-66·4	-66·4	-66·4	-66·4	+33·20	

Fig. 5.

Now release joint C, and this can also be done conveniently on line 2, because of the symmetry of the beam and its loading, the figures will be the same as for joint B. Now comes the process of carrying-over, and accepting the fact without proof for the moment, it can be shown that a moment M applied at one end of a beam, Fig. 6, will induce a moment of $\frac{1}{2}$ M, and of the opposite sign, at the other end, this moment being required to keep the beam horizontal. The carry-over, from both joints, is performed in line 3 the arrows indicating the carry-overs. (It is convenient

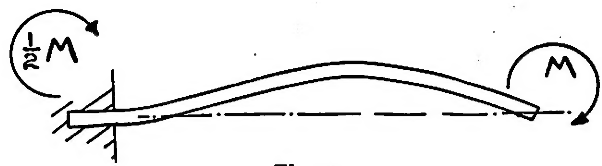


Fig. 6.

to rule a line after each distribution). Now a second distribution is carried out, line 4, B and C being released simultaneously as before. At this stage, however, there comes the question of how to treat the moments at the fixed ends. If an infinitely short extension of beam is imagined as existing to the left of A, then the moment, + 25 in this case, would be distributed as O to the right of A, and 25 to the left. Thus any moment at the end B of AB, obtained by distribution at B, has half carried over, with the sign changed to A, and after a hypothetical distribution at A receives no distributed portion, *i.e.*, is unchanged. Therefore, moments carried over to a fixed end are left there, unchanged in value and sign, and no carry-over from fixed ends (back to B in this case) can take place.

Continuing, lines 5, 6, 7 and 8 show successive distributions and carry-overs, till a point is reached where they only affect the numerical values of the final moments by negligible amounts. The process can, of course, be terminated at any point, according to the degree of accuracy required, *i.e.*, after two distributions, line 4, the moment at B is 62·5; after four distributions, line 8, the moment is 66·4, and after an infinite number of distributions the moment would be 66·66, so that a continuation of the process after four or five distributions is unwarranted, as affecting only the figures after the decimal point. In general design work no more than this number of distributions should be performed. (In this particular case, because of the symmetry of both structure and loading, it will be noted that the moments at the joints form geometric, and therefore convergent series,

$$i.e., \quad 25 + \frac{25}{4}\ (=6{\cdot}25) + \frac{25}{16}\ (=1{\cdot}56) + \frac{25}{64}\ (={\cdot}39) + \text{etc.}).$$

A final point in regard to this example, and one applicable to all examples, is that a last carry-over to the fixed ends should be performed, as in line 8, before the moments are added up, as at line 9, to give the final equilibrium moments.

Stiffness—Free Ends.

The stiffness of a beam or member of a framework is defined as $K = I/l$, and moments distributed at a joint are apportioned to

the members meeting at a joint in the ratio of their stiffnesses, as shown in Appendix A. These stiffnesses, expressed as the simplest whole-number ratio, should be written over each joint for use in the distribution process.*

Free ends, or simply-supported ends, can have no bending moment, and accordingly any moment carried over to a free end must be balanced by writing an equal moment of opposite sign below it. Then half of this moment, with the sign reversed, must be carried back to the adjacent joint. Fig. 7 shows this process applied to the same beam as in the previous example, A and D now being simply-supported ends.

Now, as a result of these steps, there is a moment of $-50+12{\cdot}5 = -37{\cdot}5$ at the end B of AB, *i.e.* of $\frac{3}{4}$ of -50, and this figure of $\frac{3}{4}$ will be same whatever the values of the moments, since $\frac{1}{2}$ of the moment is carried over, and $\frac{1}{4}$ carried back. Thus it would save labour if, instead of carrying over to A etc., the moment of 100 at B had been distributed so that end B of AB only received $\frac{3}{4}$ of its normal balancing moment, *i.e.* $\frac{3}{4}$ of $\frac{1}{2}$ the moment to be distributed, $\frac{3}{8}$, so that end B of BC would receive $\frac{5}{8}$. Thus the stiffness of AB might be taken as being $\frac{3}{4}$ of its actual stiffness K, as given by I/l. This "effective" stiffness is used in Fig. 8, where the whole example is reworked, using these modified "distribution factors" or stiffness ratios.

As a further example, consider Fig. 9, a three-span continuous beam, with end A freely-supported, and end D fixed. The stiffnesses are given over the spans to which they apply. Thus, at B,

$$\frac{K_{AB}}{K_{BC}} = \frac{13{\cdot}33}{40} = \tfrac{1}{3}, \text{ and the distribution}$$

factors are $\frac{1}{4}$ and $\frac{3}{4}$.

It is easier to treat the free end as such, however, and use the equivalent stiffness of AB. The ratio will then be

$$\frac{\frac{3}{4} \times 13{\cdot}33}{40} = \tfrac{1}{4},$$

and the distributive factors $\frac{1}{5}$ and $\frac{4}{5}$.

The F.E.M.s are either worked out, in simple cases, or may be taken from the list in Appendix C. Then, before the distribution proper, the moment at the free end must be distributed and carried over as in line 2. Then the distributions and carry-overs are repeated as often as desired, the columns being totalled algebraically to give the final moments. Knowing the support moments, they can be combined with the dead-load moments to give the

* In the examples in the text, the ratios are shown in circles.

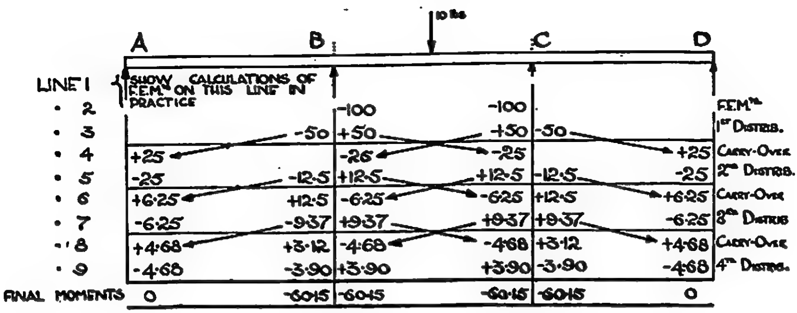


Fig. 7.

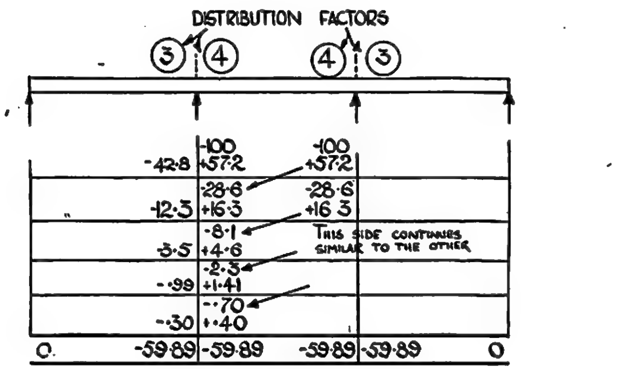


Fig. 8.

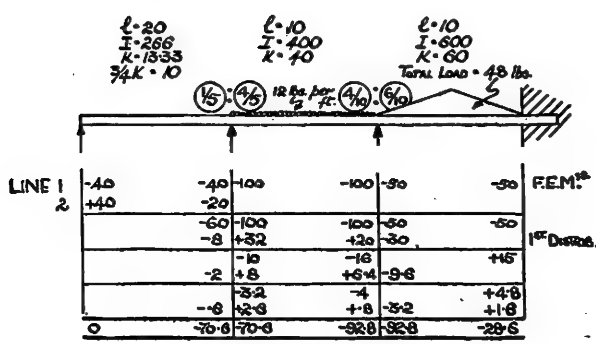


Fig. 9.

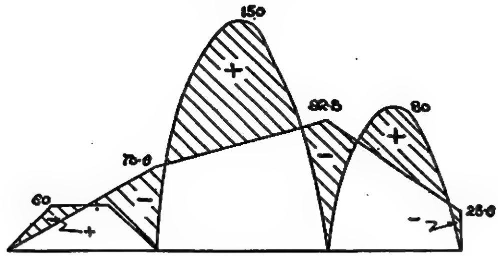


Fig. 10.

complete bending moment diagram, Fig. 10. (Note that Fig. 10 is not to scale).

Cantilever Ends.

As an example, consider Fig. 11, and imagine the fixed-end moments in AB to be 50, and the moment at B due to the load at the end of the cantilever to be 40. The unbalanced moment of 10 is to be distributed between AB and B.C. But no moment is required to rotate the end of the cantilever at B, as it has no stiffness. Thus all the unbalanced moment is distributed to end B of AB. AB can now be considered as having an equivalent stiffness of $\frac{3}{4}$ as before, and a complete example using this method is shown at Fig. 12.

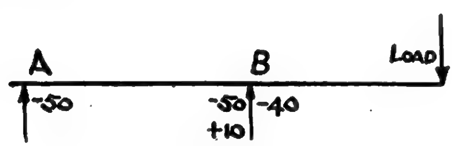


Fig. 11.

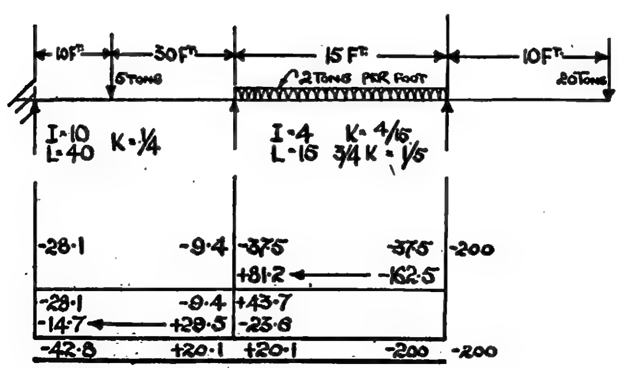
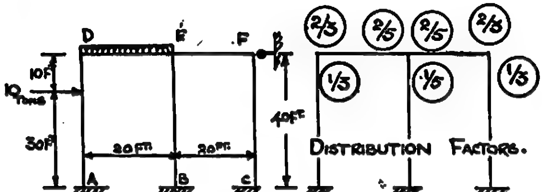


Fig. 12.

Columns.

Where a continuous beam is carried on one or more columns whose bases may be either fixed or free, the usual distribution may be carried out, although it becomes necessary to adopt a modified form of drawing up the distribution table. Two methods are given, and either may be adopted as found convenient.



F.E.M.S :- AD $\frac{10 \times 30 \times 100}{1600}$ = 18·75, 3 × 18·75 = 56·25.
DE $wl^2/12$ = 2×400/12 = 66·7.

A	D		E				F		C
	D_A	D_E	E_D	E_B	B	E_F	F_E	F_C	
-18·75	-56·25	-66·7	-66·7	0	0	0	0	0	0
	-3·48	+6·96	+26·70	-13·3		-26·70			
+1·74		-13·35	-3·48		+6·65		+13·35		
	-4·45	+8·90	+1·39	-·69	0	-1·38	-8·90	+4·45	
+2·22		-·70	-4·45	0	+·34	+4·45	+·69		-2·22
	-·23	+·47	+3·70	-1·85		-3·70	-·46	+·23	
+·12		-1·85	-·23		+·92	+·23	+1·85		-·12
	-·61	+1·24	+·18	-·09		-·18	-1·24	+·61	
+·30		-·09	-·62		+·04	+·62	+·09		-·30
+·01	-·03	+·08	+·50	-·25		-·5	-·06	+·03	-·01
-14·26	-65·05	-65·05	-43·02	-16·16	+8·08	-27·16	+5·32	+5·32	-2·78

FINAL MOMENTS

Fig. 13.

(i) The columns in the distribution table are headed by the letters of the joints, and separate columns ruled under each letter, corresponding to the members meeting at that joint. Fig. 13 shows a double span framework (or double "bent"), in which sidesway is prevented by a support at F. The distribution factors are as shown, as I is constant throughout. The distribution is carried out normally by the methods described above.

(ii) In this, the second method of writing column moments, which has the disadvantages of being less easy to write out, and is less easy to read, the moments are written at the top and bottom of any column, as shown in the simple example in Fig. 14.

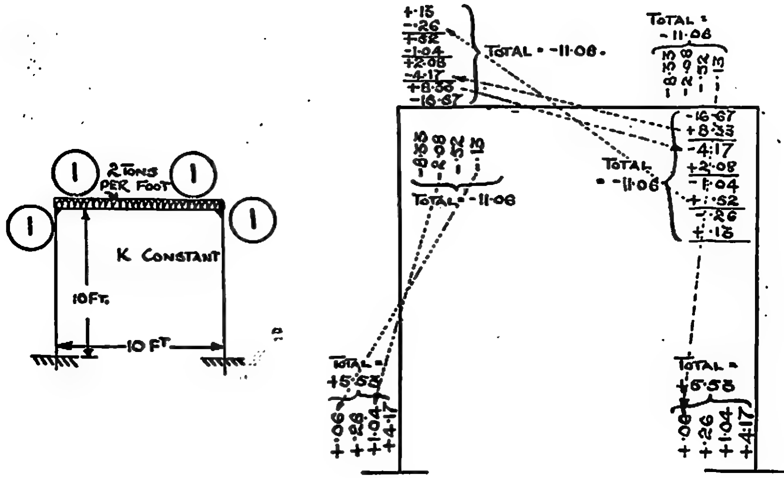

2 Tons per Foot
K Constant
10 Ft.
10 Ft.
Total = -11·06.
Total = -11·06
Total = -11·06
Total = -11·06
Total = +5·58
Total = +5·58
+·13
-·26
+·52
-1·04
+2·08
-4·17
+8·33
-16·67
+·06
+·26
+1·04
+4·17

Fig. 14.

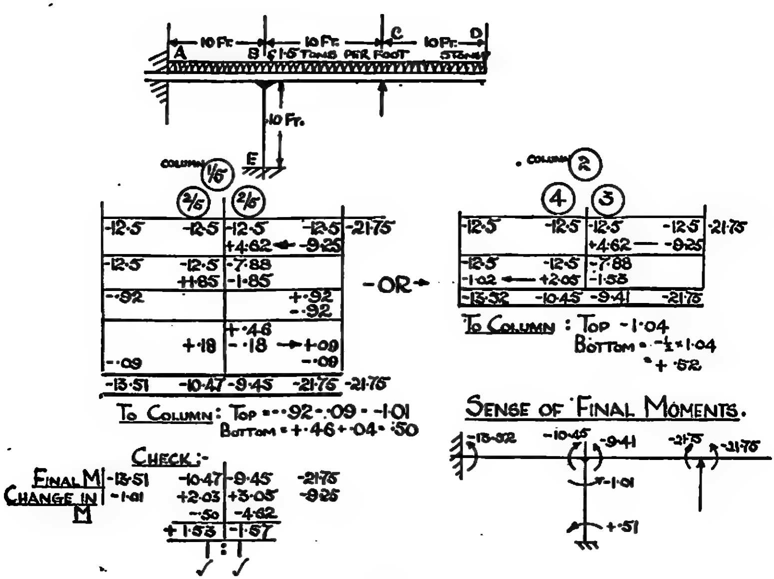

10 Ft.
10 Ft.
10 Ft.
A
B
C
D
1·5 Tons per Foot
Stone
10 Ft.
E
Column 1/5
2/5
2/5
Column 2
4
3
-OR-
To Column : Top -1·04
Bottom = -½×1·04 = +·52
To Column: Top = -·92-·09 = -1·01
Bottom = +·46+·04 = ·50
Check :-
Final M
Change in M
Sense of Final Moments.
-13·51
-10·47
-9·45
-21·75
-13·52
-10·45
-9·41
-1·01
+·51

Fig. 15.

The author considers that method (i) is preferable, especially where many columns, either of single or multi-storey type, are involved. A point to be noted in regard to columns is that it is not necessary, as has been done in Fig. 13, to carry over half the moment to the base of the column after each distribution, but as this comes more properly within the scope of abbreviated methods, it will be discussed later as a general case. Another simple example, which combines a column and a cantilever end, is shown in Fig. 15.

Check of Distribution.

At this stage it becomes necessary to enquire as to a method of checking the correctness of the distribution. It is shown in Appendix A that, at any joint, the change in moment at the joint end of a member plus half the change in moment at the other end, is proportional to the stiffness of the member. The check has been applied in Fig. 15. Another example of the use of the check is shown in Fig. 16 where the calculations for the F.E.M.s and the actual distribution are left as an exercise for the reader.

One point to be noted in regard to the use of the check, as will be seen in Fig. 16, is that the *original* F.E.M.s and stiffnesses must be used, *not* the modified stiffness due to a free end, *nor* the F.E.M.s adjusted to allow for a free end, either by itself or coupled with a cantilever end.

Sinking of Supports.

Where settlement of a support occurs, then the moments due to this settlement, assuming the joints held fixed and horizontal, may be calculated from the formulae given in Appendix A, *i.e.*,

$$M = \frac{6EI\delta}{l^2} \text{ (for a fixed-ended beam), etc.}$$

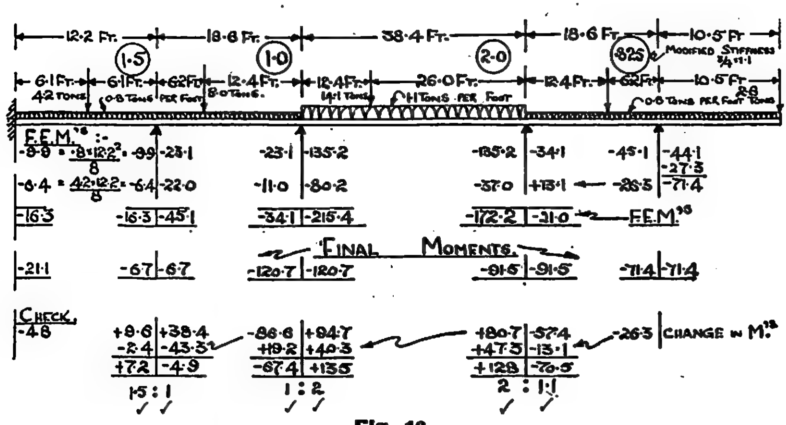
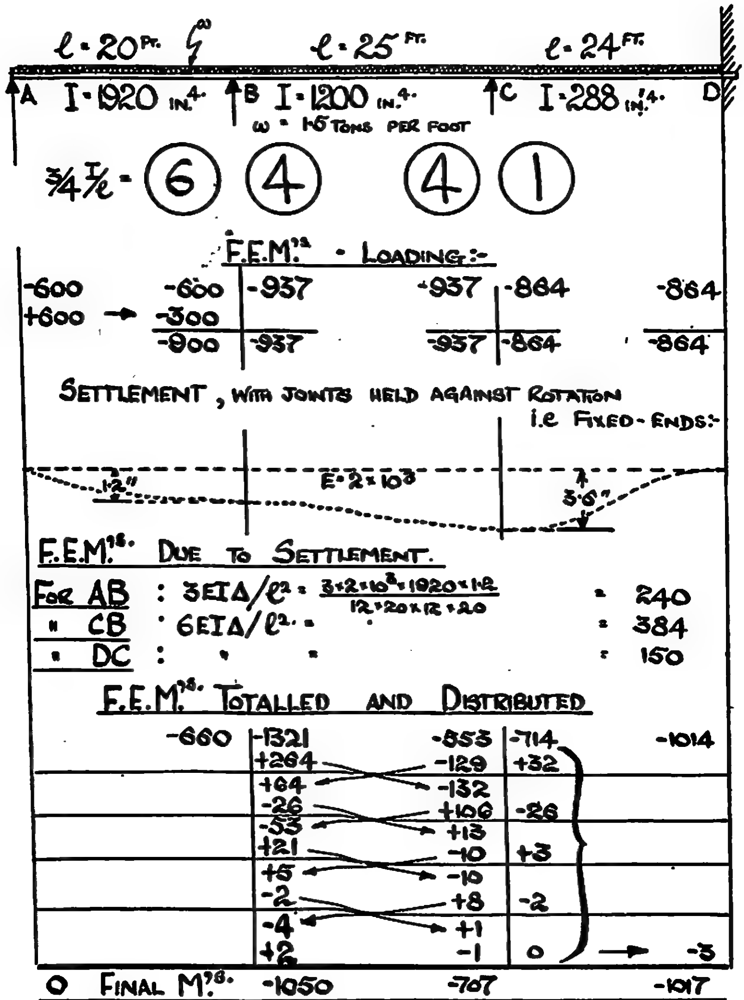


Fig. 16.



l = 20 FT.
l = 25 FT.
l = 24 FT.
A I = 1920 IN.4
B I = 1200 IN.4
C I = 288 IN.4
D
ω = 1·5 TONS PER FOOT
¾ I/l = 6 4 4 1
F.E.M.S - LOADING:-
SETTLEMENT, WITH JOINTS HELD AGAINST ROTATION
i.e FIXED-ENDS:-
E = 2 × 10³
F.E.M.S DUE TO SETTLEMENT.
FOR AB
CB
DC
240
384
150
F.E.M.S TOTALLED AND DISTRIBUTED
FINAL M.S
-1050
-707
-1017

Fig. 17.

This is done in Fig. 17, where the fixed-end moments due to the loading are first calculated, then the moments due to settlement, the two being added algebraically before the distribution is commenced.

The formulae can be applied to any case where known movements of a joint occur, as, for example, in the cases of the movements shown in Figs. 18, 19 and 20.

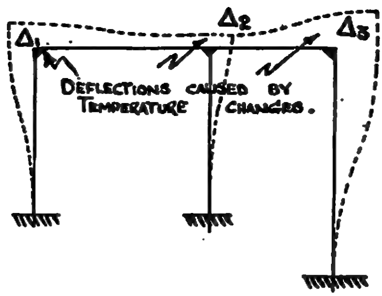


Fig. 18.

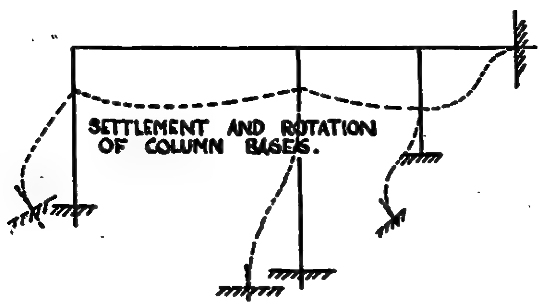


Fig. 19.

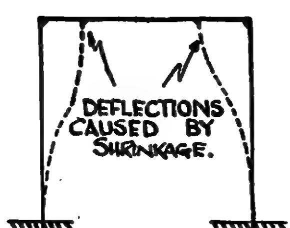


Fig. 20.

For a ready indication of the convenience and rapidity of the distribution method, as compared with the theorem of three moments, the reader is recommended to solve the example of Fig. 7, using this theorem, and contrast it with the extremely short working given in the text.

Abbreviated Methods.

After a period of use, especially where one particular type of structure only has been investigated, it becomes possible to abbreviate the method. The use of the modified stiffness for a free end ($\frac{3}{4}$ K) is the first short method, and a second one consists in taking an effective stiffness of $\frac{1}{2}$ I/L for a symmetrical interior span, where the structure and loading are both symmetrical. This is shown in Fig. 21, and it will be noted that there is (*a*) no need to carry over within the interior span concerned, and (*b*) the distribution need be performed for only half the structure, as it will be identical for the other half.

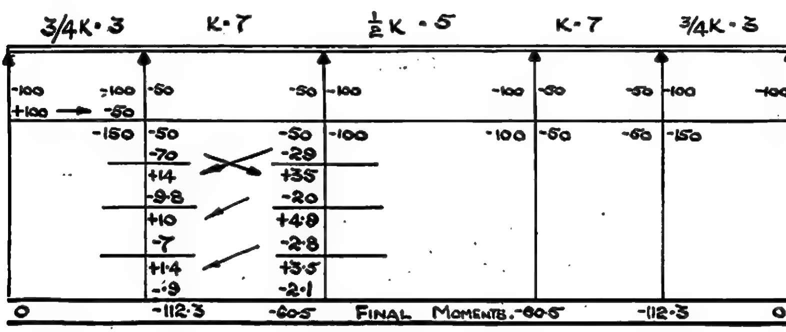


Fig. 21.

Another form of abbreviated computation consists in omitting the distributed moments, writing only the moments carried over. After the required number of distributions, the original fixed-end moments and the moments carried over are added, and the unbalanced total is then distributed. Fig. 22 is an example of this method worked in normal fashion (*a*), then using the process just described (*b*), and finally using this abbreviated form with equivalent K.s for the end spans (*c*). The only disadvantage of this shortened form is that the calculation no longer follows the physical conception of distribution.

Fig. 23 shows the method applied to the beam of Fig. 16. Two other devices which save labour are shown in Fig. 24 and in Fig. 15. In the former, the carry-overs to a fixed end are not performed after each distribution, but in one operation after the final distribution, and the latter case is similar, where moments distributed

1		1		1		
-100	-100	-50	-50	-10	-10	F.E.M.s
+100	+25	-25	+20	-20	+10	
-12.5	-50	-10	+12.5	-5	+10	
+12.5	+20	-20	-8.7	+8.7	-10	(a)
-10	-6.2	+4.3	+10	+5	-4.3	
+10	+5.2	-5.2	-2.5	+2.5	+4.3	
-2.6	-5	+1.2	+2.6	-2.1	-1.2	
+2.6	+3.1	-3.1	-2.3	+2.3	+1.2	
0	-107.8	-107.8	-18.6	-18.6	0	

-100	-100	-50	-50	-10	-10	F.E.M.s
-12.5	-50	-10	+12.5	-5	+10	Carry-overs
-10.0	-6.2	+4.3	+10.0	+5	-4.3	(b)
-2.6	-5	+1.2	+2.6	-2.1	-1.2	
-125.1	-161.2	-54.5	-24.9	-12.1	-5.5	Add.ns
+125.1	+53.3	-53.3	+6.4	-6.4	+5.5	Dist.ns
0	-107.9	-107.8	-18.5	-18.5	0	Final M.s

-100	-100	-50	-50	-10	-10	
+100 →	-50			-5	← +10	
	-150	-50	-50	-15		
		-10	+28.6			
		+8.2	-2.8			(c)
		-·8	+2.4			
	-150	-52.6	-21.8	-15		Add
	+41.5	-55.9	+3.9	-2.9		Distribute
	-108.5	-108.5	-17.9	-17.9		Final M.s

Fig. 22.

1.5		1.0		2.0		·825		
-16.3	-16.3	-45.1	-34.1	-215.4	-172.2	-21.0	-71.4	
		+30.2	-5.8	-53.5	-60.4			Carry-overs
		+8.0	+6.0	-21.3	-15.9			
		+4.6	+1.6	-5.6	-9.1			
		+1.2	+0.9	-3.2	-2.4			
		+0.7	+0.2	-0.8	-1.3			
		+0.2	+0.2	-0.4	-0.3			
-16.3	-16.3	-0.2	-31.0	-300.2	-261.6	-21.0	-71.4	Add
-4.8	+9.7		-89.7		+170.1			Distribute
-21.1	-6.6	-6.6	-120.7	-120.7	-91.5	-91.5	-71.4	

Fig. 23.

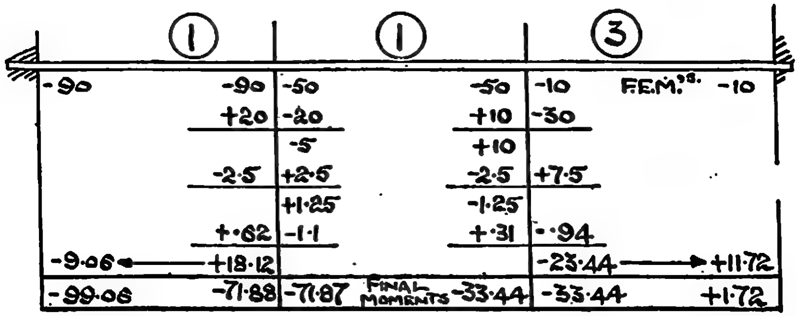


Fig. 24.

to the top of a column need not be written in a column on the distribution sheet, but obtained as the difference between the moments at the ends of the beams meeting at the column, and the moment at the base of this column (if built in) is then half of the moment at the top and of opposite sign.

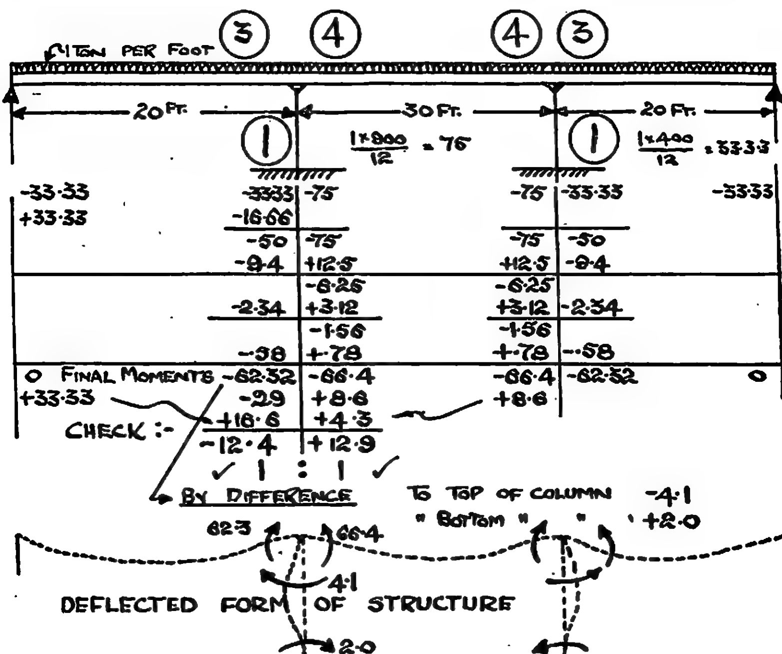


Fig. 25.

Summary of Method.

(*a*) Take fixing moments or F.E.M.s as $-^{ve}$, and therefore any moments of opposite sense as $+^{ve}$.

(*b*) Work out the stiffness or I/l value for each member meeting at a joint, using $\frac{3}{4}$ of the actual I/l value for any member whose other end is free. Reduce these stiffnesses to the lowest whole-number ratio, to give the distribution factors.

(*c*) With cantilever ends, distribute all the unbalanced moments acting at the end of the cantilever to the next span, and treat this span as free-ended.

(*d*) Carry the distributions as far as is necessary to give the requisite accuracy.

(*e*) Check the distribution after the final moments have been totalled up.

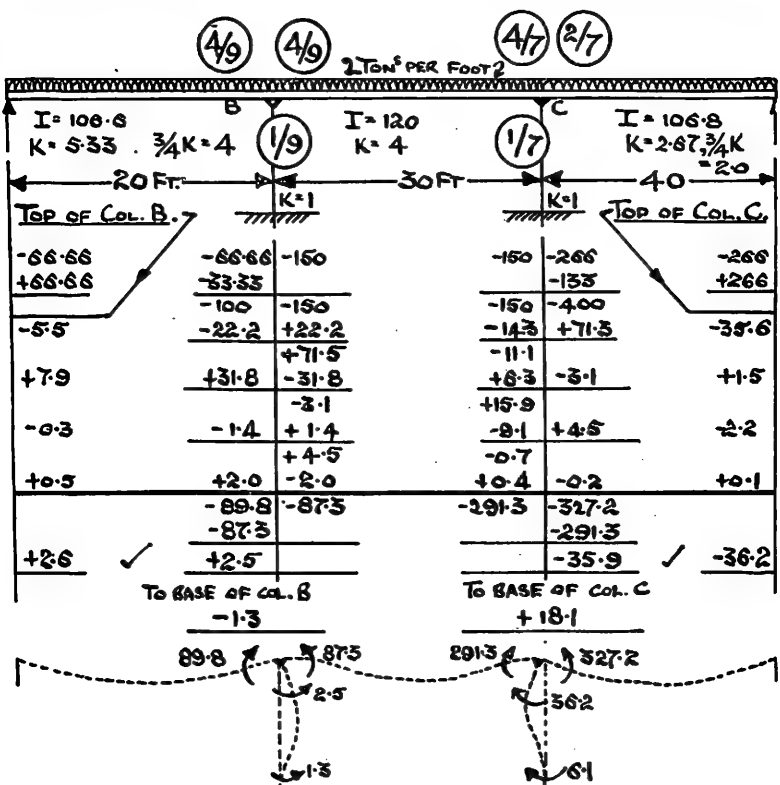


Fig. 26.

Frameworks.

The application of the method to two-dimensional frameworks follows along the lines of the examples already given, Figs. 13 and 14 being cases of simple frameworks, symmetrical both with regard to configuration and to load, or in which "sway" is prevented by other means. In general, if this symmetry exists, or if sway is prevented, then the problem is best solved by unfolding the frame, on the lines of Fig. 13, and distributing normally. Further examples are given in Figs. 25 and 26.

In practice, however, frames will be found which are neither symmetrical of outline nor of loading, and it can be visualised that

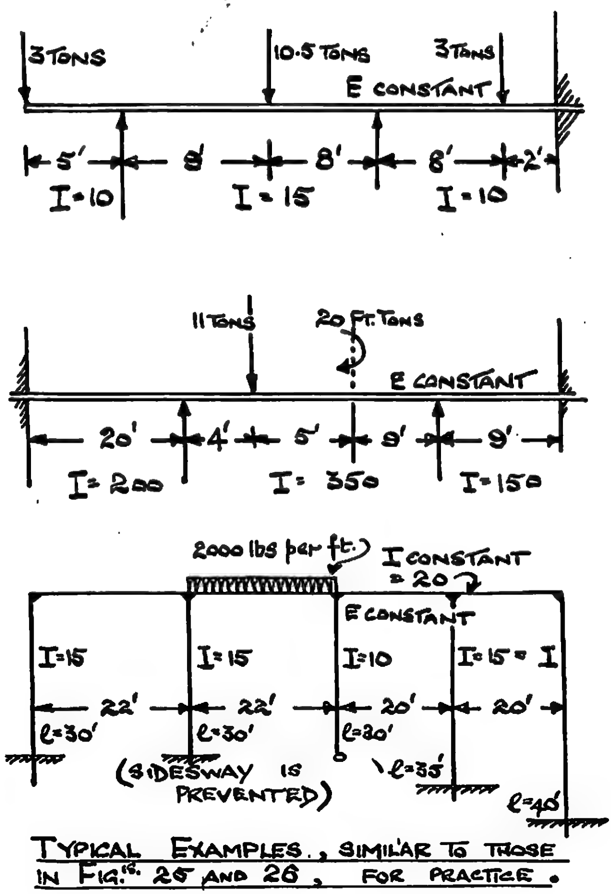


Fig. 27.

"sway" of the frame will occur, so that an amplified method of solution is necessary. Summarised, this is as follows :—

(1) Perform the moment distribution, using the F.E.M.s calculated from the loading.

(2) Calculate the unbalanced shear acting on the frame due to the moments found in (1). This shear has now to be corrected.

(3) Apply a set of arbitrary moments consistent with the geometrics of the outline, and with the structural values of the members, and distribute these moments.

(4) Deduce the unbalanced shear caused by these moments.

(5) Apply a correcting shear to the frame, which, using (4) and (3) will be in the form of moments.

(6) Total the moments due to the loading and the correcting moment given by (5) to give the final moments on the frame.

A simple example of this method, worked in detail, is shown in Fig. 28, the various steps being :—

(1) Calculate the F.E.M.s. and distribute as usual (Fig. 28 a).

(2) The sense of the distributed moments is as at Fig. 28 b, so that the unbalanced shear is 6·01 to the left.

(3) A sway as shown at (c) produces a deflection at the tops of both of the columns. Then, if the deflection is Δ, the moments on the left-hand column AB will be

$$\frac{6\,E\,I_{AB}\,\Delta}{l^2_{AB}}$$

and will be given an arbitrary value of 80.

The moment at C in CD will be $\dfrac{3\,E\,I_{CD}\,\Delta}{l^2_{CD}}$ (1)

Now, $\dfrac{6\,E\,I_{AB}\,\Delta}{l^2_{AB}} = 80$ (2)

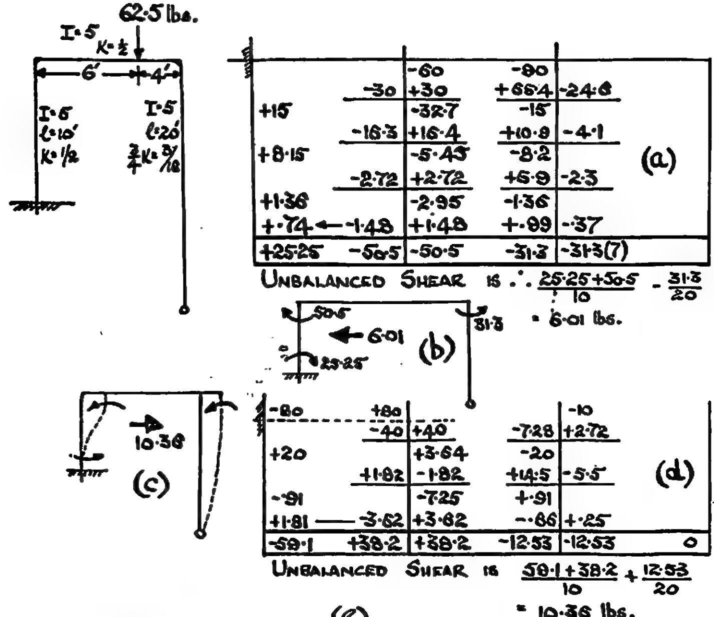
so that (1), substituting for Δ from (2), and for the given values of I_{AB}, I_{CD}, l_{AB} and l_{CD}, will be 10.

(4) These moments are distributed at (*d*) and produce a shear of 10·36 to the right.

(5) Now a correcting shear of 6·01 to the right is required, so that the moments in (*d*) are each multiplied by

$$+\frac{6{\cdot}01}{10{\cdot}36} \quad \text{and}$$

(6) added to the moments at (*a*) to give the final moments as at (*e*).



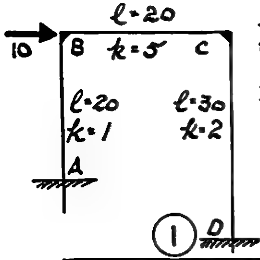
(e)

M^s (a).	+25·25	−50·5	−31·3	0
M^s(b) $\times \frac{6·01}{10·36}$	−39·4	+22·3	−7·3	0
Final M^s	−14·1	−28·2	−38·6	0

Fig. 28.

Another type of problem occurs when a bent or portal frame is subjected to a sway load, which may, or may not be combined with the usual type of gravity loading. Fig. 29 is an example, where it is required to find the moments due to a sway load of 10 units as shown, assuming the horizontal member to be rigid. This assumption is necessary in all unsymmetrically loaded frames of this type, otherwise it would be possible for the tops of the columns to move through unequal distances, and no relationship between the arbitrary moments applied to the columns would then exist. In the majority of factory buildings or other framed structures, the dimensions of the girders and columns are such that this assumption is warranted.

Arbitrary moments of 100 are first assumed for AB, and this gives arbitrary moments in CD of 133. These are distributed, and lead to a shear of 16·4 units in the same direction as the applied load of 10 units. Thus the moments given by this distribution, reduced in the ratio 10 : 16·4, will be the final moments at the various points of the frame.



Assume arbitrary moments in AB of 100

i.e. $6E\,k_{AB}\,\delta/l_{AB}=100$ ·

Then $\delta=100\,l_{AB}/6E\,k_{AB}$

Moments in CD will be $6E\,k_{CD}\,\delta/l_{CD}$

$$=\frac{100\,l_{AB}}{E\,k_{AB}}\cdot\frac{6E\,k_{CD}}{l_{CD}}=\frac{1}{2}\cdot\frac{20}{30}\times100=133$$

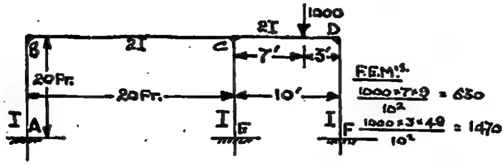
	1	5	5	2	
-100	+100	0	0	-133	+133
	-16·6	+83·4	-95	+38	
+8·3		+47·5	-41·7		-19
	+7·9	-39·6	+29·8	-11·9	
-3·9		-14·9	+19·8		+5·9
+1·2 ←	-2·5	+12·4	-14·1	+5·7 →	-2·8
-94·4	+88·8	+88·8	-101·2	-101·2	+117·1
+5·6	-11·2	+88·8	-101·2	+31·8	-15·9
	+2·8	-50·8	+44·4	-7·9	
	-8·4	+38·2	-56·8	+23·9	
	✓ 1 :	5 ✓	✓ 5 :	2 ✓	

UNBALANCED SHEAR $=\frac{94{\cdot}4+88{\cdot}8}{20}+\frac{101{\cdot}2+117{\cdot}1}{30}$

$=16{\cdot}4$

Fig. 29.

As a further example of lack of symmetry leading to the sway" type of problem, Fig. 30 may be considered. The F.E.M.s are calculated, and then distributed as at (*a*). The unbalanced shear resulting is calculated from these moments, and is seen to be 17·8 units to the right. Then, at (*b*), arbitrary moments of 100 units are applied to the columns, the fact that the columns are identical, making the moments the same for each column. These moments, on distribution, lead to an unbalanced shear of 6·5 units. Thus, these moments at (*b*), multiplied by $-17{\cdot}8/6{\cdot}5$ will be the correcting moments, and are added to the original moments at (*c*) to give the final moments on the frame. The deflected form can now be sketched, (*d*). The final bending moment diagram then follows, the free-end moments being required for CD. In this connection, and especially where reinforced concrete structures are under consideration, it is advantageous to plot moment diagrams in such a fashion that the various areas of the diagram are always on the tension side of the member. This is extremely useful when the provision and disposition of reinforcement is to be detailed.



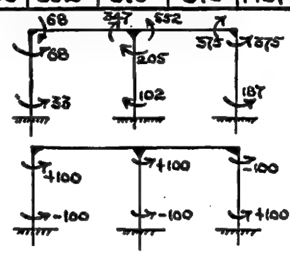
(a)

A	B_A	B_C	C_B	C_E	C_D	D_C	D_F	F
					-630	-1470		
			-180	-90	+360	+1176	-294	
		+90			-588	-180		+147
	+30	-60	-168	-84	+336	+144	-36	
-15		+84	+30		-72	-168		+18
	+28	-56	-30	-15	+57	+134	-34	
-14		+15	+28		-67	-28		+17
	+5	-10	-28	-14	+53	+22	-6	
-2		+14	+5		-11	-27		+3
-2	+5	-9	-4	-2	+10	+22	-5	+2
-33	+68	+68	-347	-205	-552	-375	-375	+187

UNBALANCED SHEAR

$= \frac{68+33+375+187-205-102}{20}$

$= 17.8 \longrightarrow$



(b)

A	B_A	B_C	C_B	C_E	C_D	D_C	D_F	F	E
-100	+100			+100			-100	+100	-100
	-33	+67	-28	+14	+58	-80	+20		+7
+16		+14	-33		+40	-29		-10	
	+5	-9	+21	+10	-42	+23	-6		-5
-2		-10	+4		-11	+21		+3	
+2	-4	+6	-4	-2	+9	-14	+7	-3	+1
-84	+68	+68	-40	+92	+54	-79	-79	+90	-97

UNBALANCED SHEAR $= \frac{68+84+79+90-92-97}{20}$

(c) $= 6.5 \longrightarrow$

-33	+68	+68	-347	-205	-552	-375	-375	+187
+230	-183	-183	+109	-251	-148	+216	+216	-246
+197	-115	-115	-238	-456	-700	-159	-159	-59

Fig. 30.

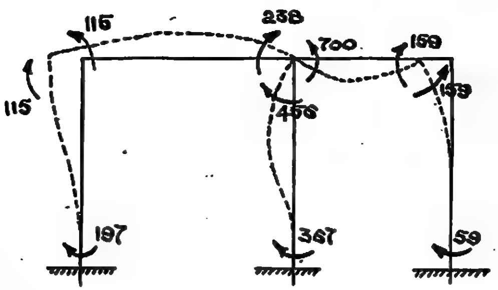
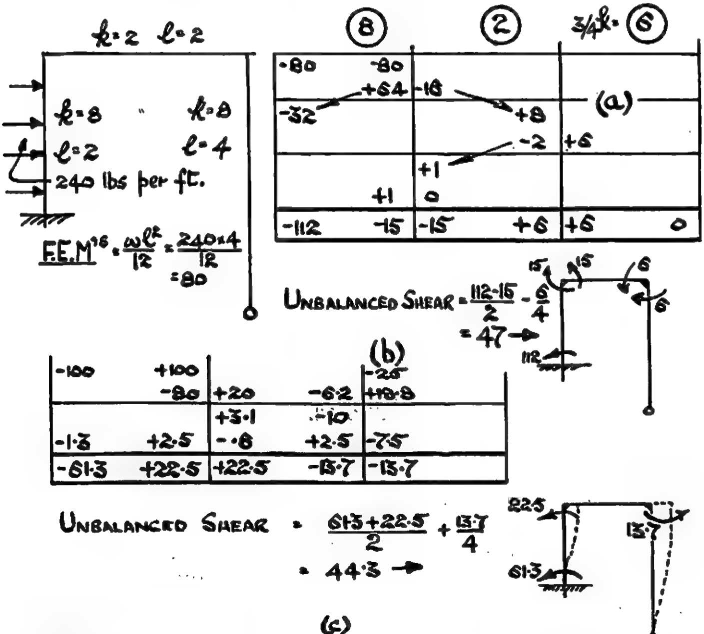


Fig. 30d.

Wind Loading.

Fig. 31 shows a single unsymmetrical portal subjected to a wind load along one side. The fixed-end moments are distributed at (*a*), giving an unbalanced shear to the right of 47. Arbitrary moments of 100 are taken as acting on AB, and those on CD will therefore be



(c)

ORIGINAL M.s	-112	-15	+6	0
CORRECTING M.s i.e. (b) × -47/44.3	+63.8	-23.6	+14.2	0
FINAL M.s	-48.2	-38.6	+20.2	0

Fig. 31.

$$\frac{100 \times K_{CD}/l_{CD} \times 6}{K_{AB}/l_{AB} \times 3} = \frac{100 \times 4 \times 2 \times 6}{8 \times 8 \times 3} = 25$$

These are distributed at (*b*), and give rise to an unbalanced shear to the right of 45·5. Thus the correcting moments to be added to those of the initial distribution are those at (*b*) multiplied by

$$-\frac{47{\cdot}0}{45{\cdot}5}$$

The final step in the calculation is then as at *c*.

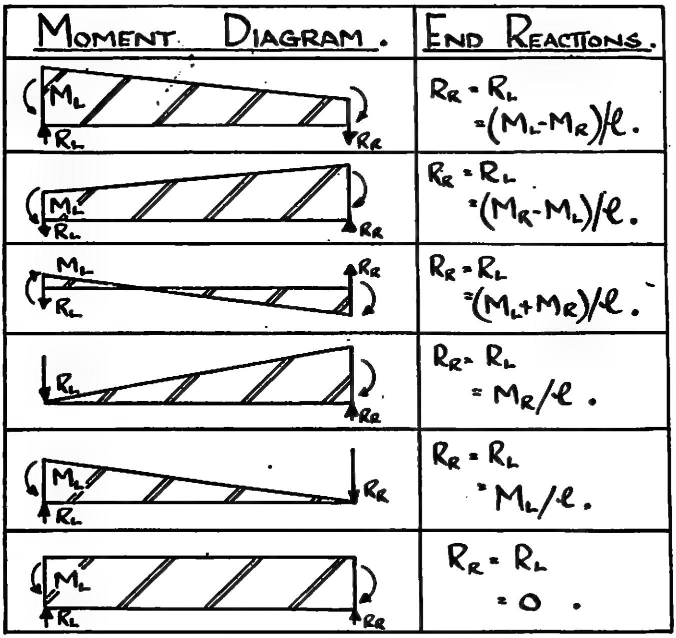


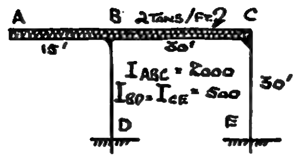
Fig. 32.

End Shears.

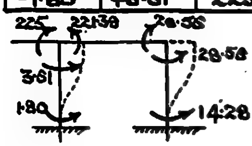
So far, no mention has been made of the method of calculating the shears on a framework from a knowledge of the moments acting on the various members. This, however, follows from the ordinary laws of statical equilibrium, and should present no difficulty. For easy reference, the various combinations of moments which may act on a member, and the resulting shears, have been tabulated in Fig. 32.

An example involving a cantilevered member is shown in Fig. 33. This is a type of structure such as might occur in a transport shelter or shopping arcade.

F.E.M.[s.]
BA : $2 \times 15^2/2 = 225$
BC : $2 \times 30^2/12 = 120$



D	B			C		E
	B_D	B_A	B_C	C_B	C_E	
0	0	−225	−150	−150	0	0
	+15		−60	+120	−30	
−7.5			−60	+30		+15
	−12		+48	−24	+6	
+6			+12	−24		−3
	+2.4		−9.6	+19.2	−4.8	
−1.2			−.96	+4.8		+2.4
	−1.92		+7.68	−3.84	+.96	
+.96			+1.92	−3.84		−.48
	+.38		−1.54	+3.07	−.77	
−.19			−1.54	+.77		+.39
	−.31		+1.23	−.62	+.15	
+.16			+.31	−.62		−.08
−.03 ←	+.06		−.25	+.32	−.10 →	+.05
−1.80	+3.61	−225	−221.39	−28.56	−28.56	+14.28

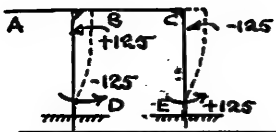


UNBALANCED SHEAR

$$= \frac{3.61 + 1.80}{30} + \frac{14.28 + 28.56}{30}$$

$= 1.608$ TONS.

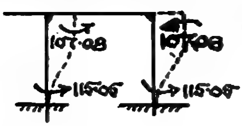
Fig. 33.



ARBITRARY MOMENTS.

D	B: B_D	B: B_C	C: C_B	C: C_E	E
-125	+125 -25	+100	-100	-125 +25	+125
+12·5	+10	+50 -40	-50 +40	-10	
FINAL MOMENTS (AFTER SEVEN DISTRIB.S)					
-115·05	+107·08	+107·08	-107·08	-107·08	+115·05

(b)



UNBALANCED SHEAR $= \dfrac{107{\cdot}08 + 115{\cdot}06}{30} \times 2 = 14{\cdot}81$ TONS $\rightarrow$

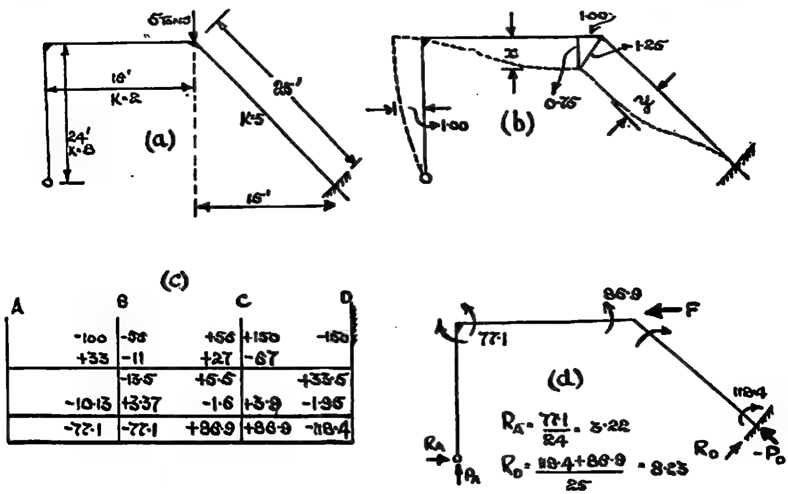
ORIGINAL MOMENTS.					
-1·80	+3·41	-225	-221·39	-28·56	+14·28
CORRECTING M.S I.E. MOM.S AT (b) × 1·608/14·81					
+12·43	-11·60	0	-11·60	+11·60	-12·43
FINAL MOMENTS					
+10·63	-7·99	-225	-232·99	-16·96	+1·85

Fig. 33 (contd.).

Frames with Sloping Members.

This type of framework is in use on the top storey of many large buildings, and can be analysed by the distribution methods already considered. Fig. 34 is a convenient example, and differs only from previous examples in that it is necessary to investigate the relative joint displacements when the arbitrary sway is applied. First of all, imagine a prop at B, acting horizontally to prevent sway. The magnitude of the reaction at the prop and the reaction at D, in the direction of DC, are found by taking moments about D and B, and are 3·75 and 6·25 tons respectively.

Now, apply an arbitrary sway to the frame as at (*b*). If AB is imagined to sway through unit distance to the left, and the joints are prevented from rotating, then C must also move through unit distance horizontally, and the movements at C are found from the small triangle of displacements shown at C. If now the moment at B in AB is given an arbitrary value of 100, those in BC and CD can be calculated as previously, expressing their deflections x and y in terms of the deflection of AB and the arbitrary moment applied to AB. These moments are now distributed as at (*c*). The remainder of the solution consists in evaluating the horizontal reaction at C from these moments (*d*), and, by comparing this



A	B		C		D
	-100	-56	+56	+150	-150
	+33	-11	+27	-67	
		-13.5	+5.5		+33.5
	-10.13	+3.37	-1.6	+3.9	-1.95
	-77.1	-77.1	+86.9	+86.9	-118.4

TAKING MOMENTS ABOUT C :- $P_A = 10.27$
" " " B :- $P_D = 18.9$
" " " D :- $F = 21.15$

ACTUAL PROPPING FORCE REQD. = 3.75, THUS CORRECTING MOMENTS ARE THOSE AT (d) × 3.75/21.15 .

Fig. 34.

reaction with that found at (*a*), calculating the correcting moments required, which are added to any original moments to give the final moments acting on the frame.

Other Fields of Application.

An example of the use of moment distribution methods applied to other than strictly structural work is Fig. 35, which represents a system of steel pipes fixed by heavy flanges to a rigid floor which is unaffected by stress or temperature. The pipes are all of the same section, and it is required to find the outside diameter of the piping if the bending stresses due to a temperature rise of 100°C. caused by passing hot oil through the system, is not to exceed 2 tons per sq. in. $E = 13{,}000$ tons per sq. in. and the coefficient of linear expansion is $11 \times 10^{-6}/°C$.

If CE is assumed to remain vertical, then the vertical deflection of D is $\triangle = 6{\cdot}6/10^2$ inches, relative to C, and the moments induced in CD, with ends constrained against rotation, are

$$\frac{6\,E\,I\,\triangle}{l^2} = \frac{6 \times 13{,}000 \times I \times 6{\cdot}6}{120 \times 120 \times 10^2} = \frac{357}{1000}\,I \quad \text{(Fig. 35 (a))}$$

Now, considering the horizontal deflection of B relative to C, this will $= 13{\cdot}2/10^2$ ins.,

and the moments (Fig. 35 (b)) are similarly $\dfrac{714}{1000}$ I.

There will be an identical horizontal deflection of D relative to C, but as DF is of different length than AB, the induced moments will be different and in the ratio of the factors $1/l^2$ for the two members, *i.e.*, the moments in DF will be

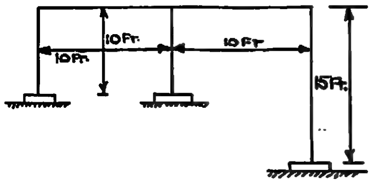
$$\frac{714\,I}{1000} \times \frac{1}{1{\cdot}5^2} = \frac{318}{1000}\,I. \qquad \text{(Fig. 35 (c))}$$

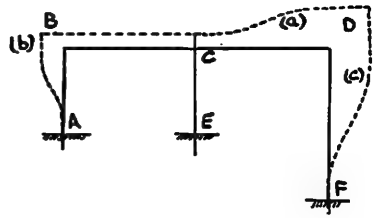
Ignoring for the moment the I/1000 factor, these moments can be given the appropriate sign and distributed (*d*).

To correct the resulting unbalanced shear, arbitrary moments are now applied :—

$$\left.\begin{aligned} M_A &= 100 = M_{CE} \\ \therefore\quad M_{BA} &= 100 = M_E \\ M_{DF} = M_F &= \frac{100}{(1{\cdot}5)^2} = 44{\cdot}5 \end{aligned}\right\} \begin{array}{l}\text{All these are} \\ \times\ \dfrac{I}{10^3}\ \text{as with} \\ \text{the F.E.M.s.}\end{array}$$

These are set out for distribution at (*e*). (To form an example for practice for the reader the actual distribution process has been omitted). This leads to an unbalanced shear in the opposite



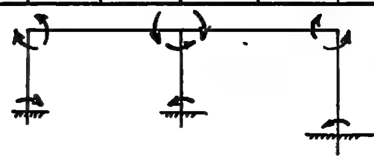


(d).

A	B (1/2) B_A	(1/2) B_C	C (1/3) C_B	(1/3) C_E	(1/3) C_D	D (3/5) D_C	(2/5) D_F	F	E
+714	-714				+357	-357	-318	+318	
	+357	-357	+119	+119	-119	+23.4	-15.6		
-178.5		-59.5	+178.5		-11.7	+59.5		+7.8	-59.5
	-29.75	+29.75	-63.4	-63.4	+63.4	-35.7	+23.8		
+14.88		+31.7	-14.88		+17.85	-31.7		-11.9	+31.7
	+15.85	-15.85	+10.98	+10.98	-10.98	+19.02	-12.68		
-7.93		-5.49	+7.93		-9.51	+5.49		+6.34	-5.49
	-2.75	+2.75	-5.81	-5.81	+5.81	-3.30	+2.20		
+1.40		+2.91	-1.38		+1.65	-2.91		-1.1	+2.91
	+1.46	-1.46	+1.01	+1.01	-1.01	+1.75	-1.16		
-.73		-.50	+.73		-.88	+.50		+.58	-.51
+.12	-.25	+.25	-.53	-.53	+.53	-.30	+.20		-.26
543.24	-372.45	372.45	+232.15	+61.5	+293.16	321.25	321.24	+319.72	-31.15

+ UNBALANCED SHEAR

$$= \frac{+543.24 + 372.45}{10} + \frac{31.15 + 61.5}{10} + \frac{319.72 + 321.24}{15} = 39.51 \longleftarrow$$



(e)

A	B_A	B_C	C_B	C_E	C_D	D_C	D_F	F	E
-100	+100			+100			-44.5	+44.5	-100
	-50	+50	-33.33	-33.33	+33.33	-26.7	+17.8		
				Etc.					
		Final		Moments.					
-77.95	+55.6	+55.6	-44.76	+77.23	+32.53	-31.18	-31.18	+27.54	-88.76

$$\text{UNBALANCED SHEAR} = \frac{77.95 + 55.6}{10} + \frac{88.76 + 77.23}{10} - \frac{27.54 + 31.18}{15} = 33.86 \longrightarrow$$

Fig. 35.

direction to the previous one, and so these moments need only be multiplied by the ratio of the two shears to give the correcting moments.

From the final moments, the maximum moment is chosen by inspection, and, with the ordinary formula for bending stresses, gives the required diameter as 8·86″, say 9″ in practice. For a comparison with the method of slope deflection, the reader is recommended to solve this example, using this latter method, and to compare the time taken with that needed in the text, remembering that the distribution need not have been carried so far in actual practice.

Conclusion.

The moment distribution method, extended and expanded from the basic principles set out above, may be employed, with rare exceptions, to all structural problems previously solved by lengthy and laborious methods, and possesses none of the disadvantages of these methods mentioned earlier.

Acknowledgments.

I am indebted to Mr. Eric Shepley, B.Sc., A.M.I.C.E., for permission to adopt two examples (Figs. 17 and 34) from his book, given in Appendix B. Also to my colleague, Mr. R. Mathieson, B.Sc., A.M.I.Mech.E., A.F.R.Ae.S., A.R.T.C., for many valuable discussions.

APPENDIX A.

The simple formulae required in the moment distribution method may be proved by the use of the slope deflection equations for a beam, the method of moment areas, or by using Castighano's theorems.

Consider a beam ABC, as shown in Fig. AI (*a*), and let all the joints be fixed. The moments will be the fixed-end moments, *i.e.* $w\,l^2/12$ for a uniformly distributed load, etc. Now, at B, the clamp or fixing must supply a moment $\overline{M}_B = M_{BA} - M_{BC}$ tending to turn the beam at the support as shown (*b*). Now, if the fixing is released at B, it will be equivalent to unloading the beam and applying a moment $\overline{M}_B$ (*c*). Let $\overline{M}_{BA}$ be the part of $\overline{M}_B$ taken by AB, and imagine that joint B rotates through an angle θ_B in an anti-clockwise fashion (*d*). Using the slope-deflection equations,

$$-\ \overline{M}_{BA} = \frac{2\,E\,I_{BA}}{l_{BA}}\ (0 - 2\,\theta_B) \qquad (1)$$

and
$$\overline{M}_{BC} = \frac{2\,E\,I_{BC}}{l_{BC}}\ (0 + 2\,\theta_B) \qquad (2)$$

Thus
$$\frac{\overline{M}_{BA}}{\overline{M}_{BC}} = \frac{I_{BA}/l_{BA}}{I_{BC}/l_{BC}} = \frac{K_{BA}}{K_{BC}} \qquad (3)$$

i.e., the unbalanced moment at B is distributed between the ends of the beams meeting at B in the proportion of their stiffnesses K $(= I/l)$.

Now $\overline{M}_B = \overline{M}_{BA} + \overline{M}_{BC}$ (ignoring the signs of the moments).

$$= \frac{K_{BA}}{K_{BC}}\ \overline{M}_{BC} + \overline{M}_{BC}$$

$$\therefore \quad \overline{M}_{BC} = \overline{M}_B\ \frac{K_{BC}}{K_{BC} + K_{BA}} \qquad (4)$$

and
$$\overline{M}_{AB} = \overline{M}_B\ \frac{K_{AB}}{K_{BC} + K_{BA}} \qquad (5)$$

These expressions, (4) and (5), give the proportions of the unbalanced moment which are taken by the beam on either side of the support, and if the proof is extended to a joint with n members meeting there, then $\overline{M}_r$, the moment taken by any member, will be given by

$$\overline{M}_r = \overline{M}\ \frac{K_r}{{}^{n}\Sigma_1 K}$$

where $\overline{M}$ is the unbalanced moment.

Carry-Over.

The physical process of clamping and releasing joints, discussed at the beginning, showed that a moment distributed to one end of a beam, resulted in a moment being transmitted or carried over to the other end, and it was assumed that ½ of the moment was carried over. This can be proved as follows :—

Consider (*d*) of Fig. A1.

Then, applying the slope deflection equation,

$$\overline{M}_A = 2\,E\,\frac{I_{AB}}{l_{AB}}\,(\theta_B) \quad = 2\,E\,\frac{I_{AB}}{l_{BA}}\;\frac{l_{AB}\,\overline{M}_{BA}}{4\,E\,I_{AB}} \qquad \text{using (1)}$$

$$= \frac{\overline{M}_{BA}}{2}$$

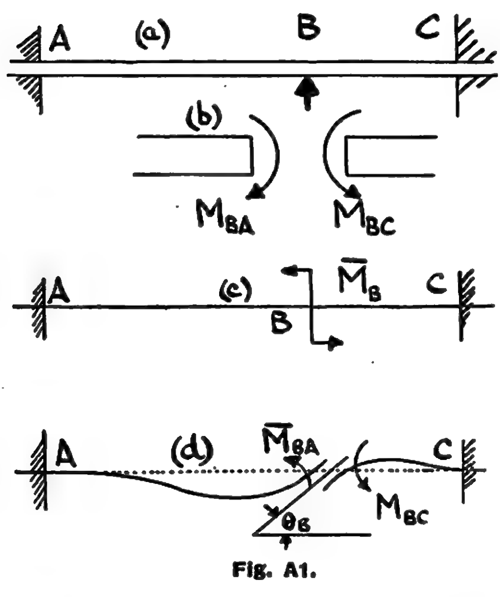


Fig. A1.

and with the sign convention used, then $\overline{M}_A$ is of opposite sign to $\overline{M}_{BA}$, as shown by Fig. 6.

Thus ½ of a moment distributed to the end of a beam at a joint, must be carried over to the other end of a beam, and its sign reversed.

Moments due to Deflection.

The moment due to the deflection of a fixed end, generally by

settlement of a support, has been taken as $\frac{6\,E\,I\,\delta}{l^2}$, or for a free end as $\frac{3\,E\,I\,\delta}{l^2}$, and this is most easily proved by the theorem of area moments.

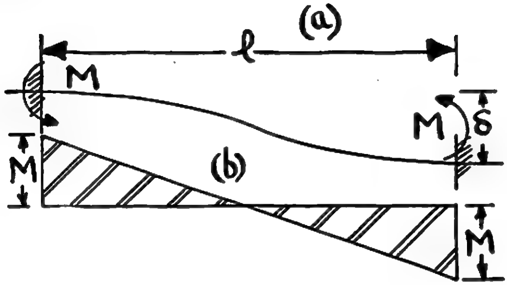


Fig. A2.

Consider a beam AB as shown, Fig. A2 (*a*), where end B is deflected downwards through a distance δ, and a moment M is required to keep the deflected end horizontal. Then the bending moment diagram is as at (*b*), and, equating the deflection of B, relative to the tangent to the beam at A, to the moment of the bending moment diagram about B, divided by E I,

$$\frac{M\,l}{4}\cdot\frac{l}{6} - \frac{M\,l}{4}\left(\frac{l}{2} + \frac{2}{3}\;\frac{l}{2}\right) = -\,E\,I\,\delta$$

$$\frac{M\,l}{4}\left(\frac{l}{6} - \frac{5}{6}\,l\right) = -\,E\,I\,\delta$$

$$\text{and } \delta = \frac{M\,l^2}{6\,EI} \quad \text{or } M = \frac{6\,E\,I\,\delta}{l^2} = \frac{6\,E\,\delta\,K}{l}$$

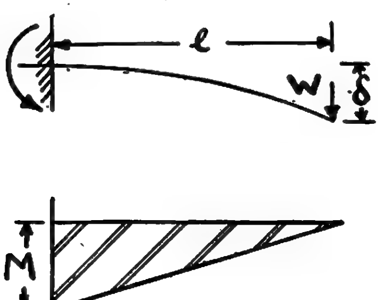


Fig. A3.

For a free end, Fig. A3,

$$M = W l$$

$$\text{and} \quad \delta = \frac{W l^3}{3\,EI}, \text{ so that } M = \frac{3\,E\,I\,\delta}{l^2} = \frac{3\,W\,\delta\,K}{l}$$

These moments are added to the fixed-end moments algebraically before the distribution commences.

Effect of Free and Fixed Ends on Distribution—General Case.

Consider a joint A, with an unbalanced torque at moment of M_A applied, and let the resulting rotation be θ_A, which is constant for all members. Then at a fixed end $\theta = O$, and at a free end $M = O$. Let members CA, BA, EA and DA, with ends as shown, Fig. A4, meet at A.

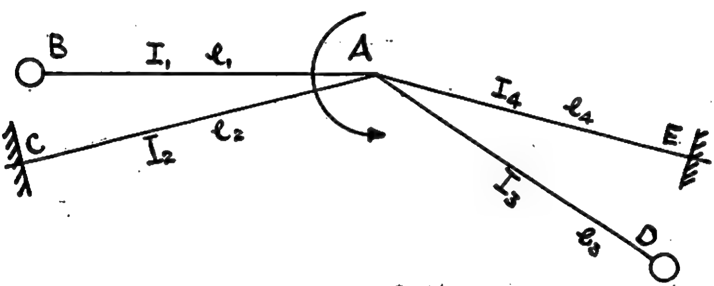


Fig. A4.

Then, using the slope deflection equations,

$$\overline{M}_B = O \text{ and } \therefore \quad -\theta_A = -2\,\theta_B$$

For Member 1, BA,

$$-\overline{M}_{A1} = \frac{2\,E\,I_1}{l_1}\left[\theta_B - 2\theta_A\right]$$

$$= \frac{2\,E\,I_1}{l_1} \times -\frac{3}{2}\,\theta_A = -\frac{3\,E\,I_1}{l_1}\,\theta_B$$

$$\text{or} \quad \theta_A = \frac{l_1}{3\,E\,I_1}\,M_{A1}$$

Member 2 (CA) :—$\theta_C = \theta$; and $-\overline{M}_{A2} = \dfrac{2\,E\,I_2}{l_2}\,(-\,2\,\theta_A$

$$\text{or} \quad \theta_A = \frac{l_2}{4\,E\,I_2}\,M_{A2}$$

Member 3 (AD) :—$M_D = 0$,

$$\theta_A + 2\,\theta_D = 0 \quad \therefore \quad \theta_D = \tfrac{1}{2}\,\theta_A$$

$$\overline{M}_{A3} = \frac{2\,E\,I_3}{l_3}\left[-\,\theta_D + 2\,\theta_A\right] = \frac{3\,E\,I_3}{l_3}\,\theta_A$$

Member 4 (AE) :— $\theta_E = 0$

$$\therefore \quad \overline{M}_{A4} = \frac{2\,E\,I_4}{l_4}\left[0 + 2\,\theta_A\right] = \frac{4\,E\,I_4}{l_4}\,\theta_A$$

Now $\overline{M}_A = \overline{M}_{A1} + \overline{M}_{A2} + \overline{M}_{A3} + \overline{M}_{A4}$, so that

$$\theta_A = \frac{\overline{M}_A}{E}\left[\frac{1}{\left\{\underbrace{3\,\Sigma\,\frac{I}{l}}_{\text{Free Ends.}} + \underbrace{4\,\Sigma\,\frac{I}{l}}_{\text{Fixed Ends.}}\right\}}\right]$$

and, for example,

$$\overline{M}_{A1} = \frac{3\,E\,I_1}{l_1}\,\theta_A = \overline{M}_A\,\frac{I_1/l_1}{\underbrace{\Sigma\,\frac{I}{l}}_{\text{Free ends.}} + \underbrace{\frac{4}{3}\,\Sigma\,\frac{I}{l}}_{\text{Fixed ends.}}}$$

and $\overline{M}_{A2}$, etc., are derived similarly.

Thus, if only one fixed- and one free-ended member meet at

$$A, \quad M_{A1} = \overline{M}_A\,\frac{I_1/l_1}{\frac{I}{l_1} + \frac{4}{3}\,\frac{I_2}{l_2}}$$

$$\text{and} \quad M_{A2} = \overline{M}_A\,\frac{I_2/l_2}{\frac{I_1}{l_1} + \frac{4}{3}\,\frac{I_2}{l_2}}$$

The above expressions give the distribution factors for members meeting at a rigid joint with a combination of fixed and free supports. The carry-over factor from a free end to a fixed end is normally $\frac{1}{2}$, but if the above distribution factors are used, in which the stiffness of fixed-ended beams is taken as 4/3 of its I/l value (or in which the stiffnesses of free-ended beams have been reduced to 3/4 of their I/l value), then the carry-over factor is zero, after one initial carry-over.

Check of Correctness of Distribution.

With the structure in its final equilibrium position, then

(i) The moments on each side of a support, in the absence of any external couple, must be equal, *i.e.* the net moment at a support must be zero.

(ii) The slopes of the beam on the two sides of a support must be equal.

Now, with fixed end moments (and any deflection moments) acting, the slope is zero before the distribution, so that the slope at a support is due to the change of moments.

Let $\triangle M_{AB}$ = Final moment—Fixed end (+deflection, if present) moments

= Change in moment due to distribution only.

and similarly for $\triangle M_{BA}$, $\triangle M_{BC}$, and $\triangle M_{CD}$.

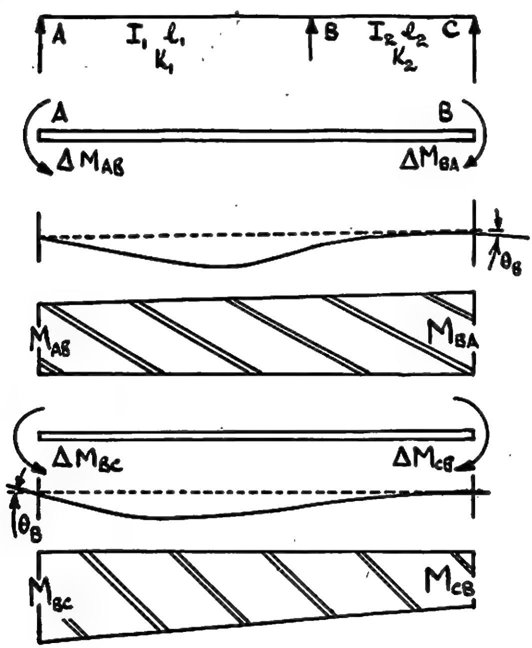


Fig. A5.

Consider Fig. A5, and taking joint B for AB, with origin at A,

$$0 = y_B - y_A = \theta_B l_1 + \frac{A\bar{x}}{E I_1} = \theta_B l_1 + \frac{l_1^2}{6 E I_1} \times [\triangle M_{AB} + 2 \triangle M_{BA}]$$

$$i.e., \ \theta_B = -\frac{\triangle M_{BA} + \frac{1}{2} \triangle M_{AB}}{3 E K_1}$$

For BC, with origin at C,

$$\theta_B = + \frac{\triangle M_{BC} + \frac{1}{2} \triangle M_{CB}}{3 E K_2}$$

$$\text{Thus} \quad \frac{\triangle M_{BA} + \frac{1}{2} \triangle M_{AB}}{\triangle M_{BC} + \frac{1}{2} \triangle M_{CB}} = - \frac{K_1}{K_2}$$

i.e., the sum of the changes in bending moment in a member (*a*) at a joint and (*b*) at the other end of a member, are proportional to the stiffness of the member.

The check has been applied to the distributions shown in Figs. 15 and 16. It will be noted that in some cases, *i.e.* Fig. 29, the check does not give the expression $\triangle M_{BA} + \frac{1}{2} \triangle M_{AB}$ as being exactly proportional to K for the member considered. This is because in these examples the distributions have only been carried to three stages, and the resulting moments are therefore only rough approximations. The reader is recommended to carry the distributions to six or seven stages for practice, and then to apply the check again.

APPENDIX B.

Further information on moment distribution methods may be obtained from Professor Hardy Cross's original paper (Trans. Amer. Soc. C.E., Vol. 96, 1932, Paper No. 1793), or his book, in collaboration with Professor N. B. Morgan, "Continuous Frames of Reinforced Concrete," John Wiley and Sons, and also from the following books and papers, among others :—

"Structural Analysis," Cassie, Longmans.

"Continuous Beam Structures," Shepley, Concrete Publications.

"Analysis of Engineering Structures," Pippard and Baker.

"Concrete and Constructional Engineering," April and July, 1932, and October, 1934.

"The Problem of Sway in Complicated Rigid Frames," J. Inst. C.E., Vol. 24 1944-5), No. 6, p. 99 (April, 1945).

"Moment Distribution Applied to Rectangular Rigid Space Frames," J. Inst. C.E., 1947-8, Jan. '48, No. 3, p. 221.

These latter works all use a modified form of the original notation used in the preceding examples, in that fixed-end moments are given as $-^{ve}$ (left-hand end) and $+^{ve}$ (right-hand end). This means that no change of sign is needed when carrying over, and involves other small modifications in the use of signs. This sign convention does not seem to possess any marked advantages over the original convention in which F.E.M.s are taken as both $-^{ve}$. However, it is recommended that practice be had with both, the final choice of convention being based on experience.

APPENDIX C.

Below are given the commonly needed F.E.M.s, in terms of the span and the loading.

All general cases have been included, from which any particular case may be derived, *i.e.* a point load dividing a beam of length l into portions of lengths a and b is considered, and the F.E.M.s given is terms of a and b and l. The case of a number of loads at various positions along the beam can then be worked out, using this result, Fig. C.1.

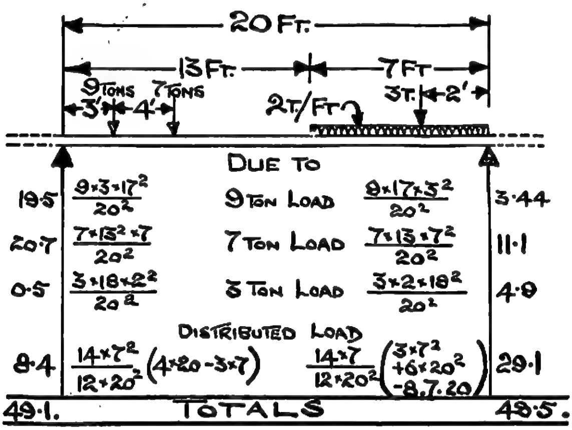


Fig. C1.

The moment distribution method itself has a wide application in the determination of fixed-end moments. The case of a beam with an unsymmetrical point load, Fig. C.2, furnishes a simple example. In this case fixed-end moments of a convenient magnitude are given to one portion, AB, of the beam. Since the deflection of the other portion is identical, the moments in this portion will be in proportion to these arbitrary moments, in the ratio of the values of I/l^2* for the two portions. These moments are distributed, and the shears on the beams computed from the final moments, and thus the load producing the arbitrary moments. The true moments are then found by multiplying these arbitrary

$$\text{moments by the ratio of } \frac{\text{actual load}}{\text{load from arbitrary moments.}}$$

* See Appendix A.

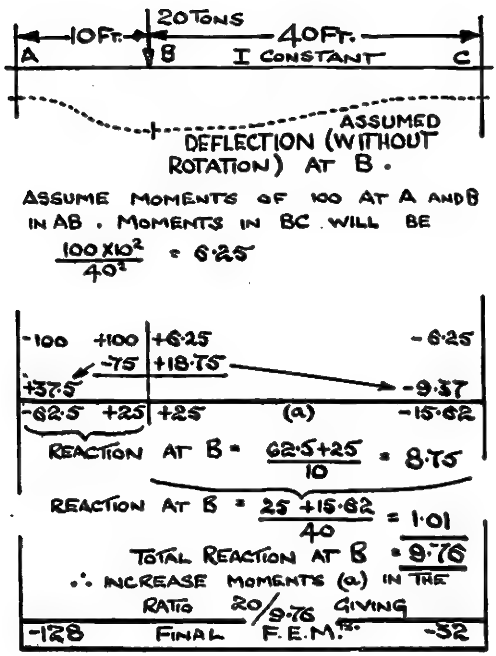

20 TONS
10 FT.
40 FT.
A
B
I CONSTANT
C
ASSUMED DEFLECTION (WITHOUT ROTATION) AT B.
ASSUME MOMENTS OF 100 AT A AND B IN AB. MOMENTS IN BC WILL BE
$\frac{100 \times 10^2}{40^2} = 6{\cdot}25$
-100
+100
+6·25
-6·25
-75
+18·75
+37·5
-9·37
-62·5
+25
+25
(a)
-15·62
REACTION AT B = $\frac{62{\cdot}5+25}{10} = 8{\cdot}75$
REACTION AT B = $\frac{25+15{\cdot}62}{40} = 1{\cdot}01$
TOTAL REACTION AT B = 9·76
∴ INCREASE MOMENTS (a) IN THE RATIO 20/9·76 GIVING
-128
FINAL F.E.M.$^{S.}$
-32

Fig. 62.

TABLE OF FIXED-END MOMENTS.

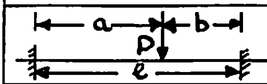
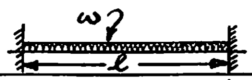
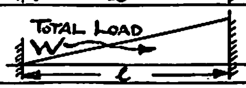
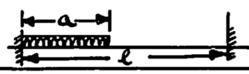
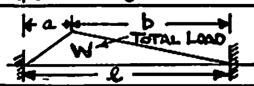
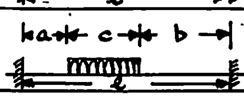
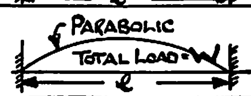
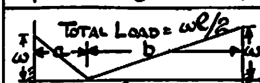
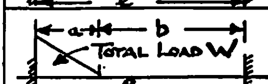
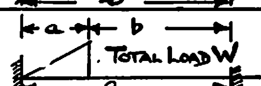
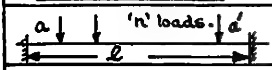
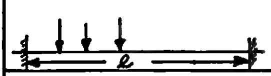
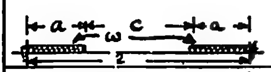
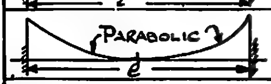
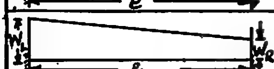
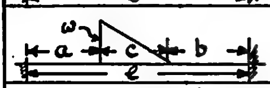
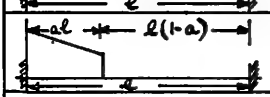
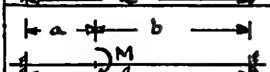
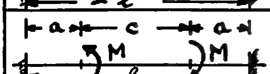
LOADING	M_L	M_R
	$P\,a^2b/\ell^2$.	$P\,b^2a/\ell^2$.
	$\omega\ell^2/12$.	$\omega\ell^2/12$.
	$\cdot 8\,\frac{W\ell}{12}$.	$1{\cdot}2\,\frac{W\ell}{12}$.
	$\frac{Wa}{12\ell^2}(3a^2-8a\ell+6\ell^2)$.	$\frac{Wa^2}{12\ell^2}(4\ell-3a)$.
	$M_L = \frac{W}{30\ell^2}\left[2a^2(a+4b)+3b^2(4a+b)\right]$. $M_R = \frac{W}{30\ell^2}\left[3a^2(a+4b)+2b^2(4a+b)\right]$.	
	W = TOTAL LOAD. $W/12\ell^2 \times \{12ab(b+c)+6cb^2+4c^2(a+b)+c^3\}$	$\times\{12ab(a+c)+6ca^2+4c^2(a+b)+c^3\}$
	$\frac{1}{10}W\ell$.	$\frac{1}{10}W\ell$.
	$\frac{\omega}{60\ell}(2\ell^3-3a\ell^2+7a^2\ell-3a^3)$.	$\frac{\omega}{60\ell}(2\ell^3-3b\ell^2+7b^2\ell-3b^3)$
	$\frac{Wa}{30\ell^2}(3a^2-10a\ell+10\ell^2)$.	$\frac{Wa^2}{30\ell^2}(5\ell-3a)$.
	$\frac{Wa}{15\ell^2}(6a^2-15a\ell+10\ell^2)$.	$\frac{Wa^2}{10\ell^2}(5\ell-4a)$.

TABLE OF FIXED-END MOMENTS (CONTD.)

LOADING.	M_L.	M_R.
	FOR A NUMBER OF EQUAL LOADS, TOTAL LOAD = P, AND $a = a' = \frac{1}{2}$ LOAD SPACING $M_L = M_R = \left(\frac{1}{12}+\frac{1}{24n^2}\right)Pl$.	
	WITH EQUAL LOADS AND ALL SPACINGS IDENTICAL $M_L = M_R = \left(1-\frac{1}{n^2}\right)\frac{1}{12}Pl$.	
	$M_L = M_R = \frac{wa^2}{6l}(3l-2a)$.	
	IF W = TOTAL LOAD $M_L = M_R = \frac{Wl}{20}$.	
	$(3w_L+2w_R)\frac{l^2}{60}$.	$(2w_L+3w_R)\frac{l^2}{60}$.
	$\left[10bc(b+c)+15a(2b^2+c^2)+40abc+3c^3\right]{}^{wc}/_{60\,l^2}$.	$\left[20ac(a+b)+5c^2(2a+b)+30a^2b+2c^3\right]{}^{wc}/_{60\,l^2}$.
	$(10-20a+15a^2-4a^3)\frac{wl^2a^2}{20}$.	$(10-15a+6a^3)\frac{a^3wl^2}{30}$.
	$\frac{b}{l}\left(2-3\frac{b}{l}\right)M$.	$\frac{a}{l}\left(2-3\frac{a}{l}\right)M$.
	$\frac{c}{l}M$.	$\frac{c}{l}M$.

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2.	Deflection of Shafts and Beams.	Connected.
3.	Deflection of Shafts and Beams (Instruction Sheet).	
4.	Steam Radiation Heating Chart.	
5.	Horse-Power of Leather Belts, etc.	
6.	Automobile Brakes (Axle Brakes).	Connected.
7.	Automobile Brakes (Transmission Brakes).	
8.	Capacities of Bucket Elevators.	
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10.	Shafts up to 5½ inch diameter, subjected to Twisting and Combined Bending and Twisting.	
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15.	Automobile Clutches (Cone Clutches).	
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17.	Coil Friction for Belts, etc.	
18.	Internal Expanding Brakes. Self-Balancing Brake Shoes (Force Diagram).	Connected.
19.	Internal Expanding Brakes. Angular Proportions for Self-Balancing.	
20.	Referred Mean Pressure Cut-Off, etc.	
21.	Particulars for Balata Belt Drives.	
22.	⅞″ Square Duralumin Tubes as Struts.	
23.	1″ „ „ „ „	
24.	¾″ Square Steel Tubes as Struts (30 ton yield).	
25.	⅞″ „ „ „ „ (30 ton yield).	
26.	1″ „ „ „ (30 ton yield).	
27.	¾″ „ „ „ (40 ton yield).	
28.	⅞″ „ „ „ (40 ton yield).	
29.	1″ „ „ „ (40 ton yield).	
30.	Moments of Inertia of Built-up Sections (Tables).	Connected.
31.	Moments of Inertia of Built-up Sections (Instructions and Examples).	
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35.	Screw Propeller Design (Sheet 1, Diameter Chart).	Connected.
36.	„ „ „ (Sheet 2, Pitch Chart).	
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38.	Open Coil Conical Springs.	
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65. Ring with Uniform Internal Load (Tangential Strain). } Connected.
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70. „ „ „ Stress. }
71. Ring with Uniform External Load, Tangential Strain. } Connected.
72. „ „ „ „ „ Stress. }
73. Viscosity Temperature Chart for Converting Commercial to Absolute Viscosities. } Connected.
74. Journal Friction on Bearings. }
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76. Shearing and Bearing Values for High Tensile Structural Steel Shop Rivets in accordance with B.S.S. No. 548/1934.
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80. (See No. 105).
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93-4-5-6-7. Rolled Steel Sections.
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103. Punch and Die Clearances for Blanking and Piercing.
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106. Compound Cylinder (Similar Material) Radial Pressure of Common Diameter (D1).
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110. Permissible Stresses in Cranes—B.S. 2573. Table 1—Values of A & B Tons/Sq. In.
111. Permissible Stresses in Cranes—B.S. 2573. Table 2—Values of A & B Tons/Sq. In.
112. Permissible Stresses in Cranes—B.S. 2573. Table 3—Compressive Stresses in Axially Loaded Struts.
113. Permissible Stresses in Cranes—B.S. 2573. Table 4—Power Driven Rivets and Fit Bolts.
114. Permissible Stresses in Cranes—B.S. 2573. Table 5—Power Driven Rivets and Fit Bolts.
115. Rectangular Weld Groups with Eccentric Load in Plane of Weld.

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